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Some notes on subspace convex-cyclicity

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Abstract. A bounded linear operator T on Banach space X is subspace convex-cyclic for a subspace M if there exists a vector $x \in M$ such that $Co(orb(T,x)) \cap M$ is dense in M. We construct examples of subspace convex-cyclic operator that is not convex-cyclic. In particular, we prove that every convex-cyclic operator on the separable Banach space X is a subspace convex-cyclic operator for some pure subspace M of X.

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1 Introduction

A linear and bounded operator T on a Banach space X is said to be hypercyclic or supercyclic if there exists a vector $x \in X$ whose orbit $orb(T,x) = \{T^n x; \quad n = 0, 1, 2, ...\}$ or the set of all scalar multiples of orbit, i.e., $\lambda orb(T,x)$, is dense in X, respectively. Also, the operator T is cyclic if for a vector $x \in X$ the linear span generated by $orb(T,x)$, i.e., $span(orb(T,x))$, is dense in X and in this case the vector x is a cyclic vector for operator T. Observe that the supercyclicity falls between

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cyclicity and hypercyclicity. It is well known that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for all pair of nonempty open subsets U and V of X, there exists a non-negative integer n such that $T^n(U) \cap V \neq \phi$, [5]. It is worthwhile to mention that if the operator T is hypercyclic, then the underlying Banach space X should be separable and on the other side, some authors showed that every separable infinite-dimensional Banach space supports a hypercyclic operator, [1], and [4]. As well as separability or non-separability of the underlying Banach space is of great importance, paying attention to the dimension of the underlying Banach space requires more precision. In 1969, Rolewicz showed that no operators on a finite-dimensional Banach space are hypercyclic, [14], from this perspective, a significant number of Banach spaces are not considered. One can see many other properties on hypercyclicity, supercyclicity and cyclicity in [2], and [7].

In 2011, the authors in [11] studied the density of the orbit in a pure subspace instead of the whole space. This phenomenon is called subspace-hypercyclicity. Subspace-hypercyclic operator shares some of the same properties as a hypercyclic operator. For example, no operators on a finite-dimensional Banach space are subspace-hypercyclic, or the spectrum, $\sigma(T)$, of every subspace-hypercyclic operator T on X meets the unit circle $\mathbb{T}$. Some references on the subspaces-hypercyclicity are [9], [10], and [12].

In 2013, Rezaei introduced the concept of convex-cyclicity and he characterized, completely, convex-cyclic operators on $\mathbb{F}^n$, $n \geqslant 1$ where $\mathbb{F}$ denotes either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$. to be precise;

Definition 1.1. An operator $T \in X$ is said to be convex-cyclic if there exists a vector $x \in X$ such that the convex hull generated by $orb(T, x)$, i.e., $Co(orb(T, x))$, is dense in X and such a vector x is said to be a convex-cyclic vector for T.

We refer to [13] for complete information on convex-cyclic operators and related properties on the finite-dimensional Banach spaces and in the following, we will focus on the infinite-dimensional separable Banach spaces. Rezaei, [13] asks some questions which one can look for their answers and some other properties in [3], [15] and [16].

In the next section, first, we define subspace convex-cyclicity and give some examples of subspace convex-cyclic operators which are not convex-cyclic. Then a Hahn-Banach Characterization will be given for subspace convex-cyclicity. In the following, we will show that every convex-cyclic operator on the separable Banach space X, is a subspace convex-cyclic operator for a non-trivial subspace M of X.

2 Main Results

In the following, X always denotes an infinite-dimensional separable Banach space over the field $\mathbb{F}$, that $\mathbb{F}$ denotes either the complex field, $\mathbb{C}$, or the real field, $\mathbb{R}$. Whenever we talk about a subspace M of X we will assume that M is a non-trivial close subspace of X and also let $L(X)$ be the set of all continuous linear operators on X. In present work, $\mathcal{CP}$ denotes the set of all convex polynomials $P(z) = \sum\limits_{i=1}^{n} c_i z^i$, $n \in \mathbb{N}$, $c_i \geqslant 0$ for all i and $\sum\limits_{i=1}^{n} c_i = 1$.

Definition 2.1. Let $T \in L(X)$ and M be a non-trivial subspace of X. The operator T is said to be subspace convex-cyclic as for M (or M-convex-cyclic), if for some vectores $x \in X$, $\overline{Co(orb(T,x)) \cap M} = M$. A vector $x \in X$ with this property is said to be an M-convex-cyclic vector.

In the following, we show that subspace convex-cyclicity is not equivalent to convex-cyclicity. Before we state the first example, recall that for the sequence $w = \{w_n \geq 0\}_{n\in\mathbb{N}}$ of positive integer and the canonical basis $\{e_n\}_{n\in\mathbb{N}}$ of $\ell_p(\mathbb{N})$ the operator B_w on $\ell_p(\mathbb{N})$ for $1 \leq p < \infty$, which is defined by;

$$B_w(e_j) = \begin{cases} w_j e_{j-1}, & j \geq 2 \\ 0 & j = 1 \end{cases},$$

is called a unilateral weighted backward shift. Rezaei in [13] gave an equivalent condition for B_w to be convex-cyclic on $\ell_p(\mathbb{N})$. In fact, the unilateral weighted shift B_w with the weight sequence $\{w_n\}_{n\in\mathbb{N}}$ is convex-cyclic if and only if

$$\limsup_n (\Pi_{i=1}^{n} w_i) = +\infty.$$

One can see some other properties of the convex-cyclic unilateral weighted shifts on $\ell_p(\mathbb{N})$ in [13].

Example 2.2. Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a convex-cyclic vector for the unilateral weighted shift B_2, $(2 = \{2, 2, ...\})$, on $\ell_p(\mathbb{N})$, $1 \leq p < \infty$. If I is the identity operator on $\ell_p(\mathbb{N})$, then it is easy to show that $x \bigoplus 0$ is a subspace convex-cyclic vector for $B_2 \bigoplus I \in L(\ell_p(\mathbb{N}) \bigoplus \ell_p(\mathbb{N}))$ with respect to the subspace $\ell_p(\mathbb{N}) \bigoplus \{0\}$ of $\ell_p(\mathbb{N}) \bigoplus \ell_p(\mathbb{N})$. Note that the operator $B_2 \bigoplus I \in \ell_p(\mathbb{N}) \bigoplus \ell_p(\mathbb{N})$ is not a convex-cyclic operator.

Example 2.3. Let $\mathcal{L} = L(H)$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the algebras of all bounded linear operators and compact operators on separable Hilbert space H, respectively. Consider the norm operator topology on $\mathcal{L}$ and $\mathcal{K}$. Then $\mathcal{L}$ is not separable while $\mathcal{K}$ is separable. Indeed, the left multiplication operator $L_T \in L(\mathcal{L})$ defined by $L_T(S) = TS$ is not hypercyclic, but $L_T \in L(\mathcal{K})$ is hypercyclic, [6]. Thus there exists an operator $S \in \mathcal{K}$ such that with respect to norm operator topology

$$\overline{orb(L_T, S)} = \mathcal{K},$$

thus

$$\overline{Co(orb(L_T, S)) \cap \mathcal{K}} = \mathcal{K}.$$

Therefore S is a subspace convex-cyclic vector for the left multiplication operator L_T. Since the norm of every convex-cyclic operator is bigger than one [13], so consider $\| T \| \leq 1$, to ensure that the left multiplication operator $L_T \in L(\mathcal{L})$ is not convex-cyclic.

A necessary and sufficient condition for an operator so as to have a convex-cyclic vector was establihed in [3]. Now we have a simillar necessary and sufficient condition for subspace convex-cyclic operatores.

Theorem 2.4 (The Hahn-Banach characterization for subspace convex-cyclicity)**.** *Let M be a non-trivial subspace and $T \in L(X)$. Then for a vector $x \in X$, the following are equivalent:*

1. *the vector x is an M-convex-cyclic vector for T.*
2. *For every non-zero functional $\Lambda \in M^*$ we have*

$$\sup Re(\Lambda(P(T)x)) = +\infty,$$

when $P \in \mathcal{CP}$ and $P(T)x \in M$.

Proof. Assume that (1) holds and $\Lambda \in M^*$ is an arbitrary non-zero continuous linear functional. Since the set $Co(orb(T,x)) \cap M$ is convex and

$$\mathbb{R} = Re(\Lambda(\overline{Co(orb(T,x)) \cap M})) \subseteq \overline{Re(\Lambda(Co(orb(T,x)) \cap M))}$$

and

$$Co(orb(T,x)) \cap M = \{P(T)x \mid P \in \mathcal{CP}\ , P(T)x \in M\},$$

so (2) is true.

Now let $y \in M$ be a vector such that is not in the closure of $Co(orb(T,x)) \cap M$. Then there exist a functional $\Lambda \in M^*$ such that $Re(\Lambda(y))$ is an upper bound for all $Re(\Lambda(P(T)x))$ when $P(T)x \in M$ and P is a convex polynomial. Thus if (2) holds, then (1) does also. $\square$

In the following we affirmative answer to this question that if an operator T on X is convex-cyclic, must there be a special subspace M of X such that T is an M-convex-cyclic operator?. First we state the following Proposition:

Proposition 2.5. *Let $x \in X$ be an M-convex-cyclic vector for $T \in L(X)$ and let X_1 be a closed subspace of X such that there exist a vector $y \in X$ with $||y|| > 1$ and $dist(y, X_1) > 1$. Then for every $z \in X_1$, there exists a sequence of convex polynomials $\{P_n(\alpha)\}$ such that $P_n(T)x \longrightarrow z$ as $n \to +\infty$ and for all n;*

$$dist(y, lin\{X_1, P_n(T)x\}) > 1.$$

Proof. Let $\varepsilon > 0$ and $Q : X \longrightarrow \dfrac{X}{X_1}$ be the natural map. Since $dist(y, X_1) > 1$, so $||Q(y)|| > 1$ and consider a nonzero continouse linear functional $\Lambda \in (\dfrac{X}{X_1})^*$ with $||\Lambda|| = 1$ and $||\Lambda(Q(y))|| = ||Q(y)||$. Clearly

$$V = \left\{Q(m);\ m \notin X_1,\ ||Q(m)|| < \varepsilon,\ \frac{|\Lambda(Q(m))|}{||Q(m)||} < \frac{||Q(y)|| - 1}{||Q(y)|| + 1}\right\}$$

is an open subset of Banach space $\frac{X}{X_1}$. Fix $Q(m) \in V$ and if $r \in \overline{B}\left(0, \frac{||Q(y)||+1}{||Q(m)||}\right)$, then

$$\begin{aligned}&||Q(y) - rQ(m)|| = ||\Lambda||||Q(y) - rQ(m)|| \geq |\Lambda(Q(y - rm))| = \\ &||\Lambda(Q(y))| - |r||\Lambda(Q(m))|| > \\ &||Q(y)|| - \left(\frac{||Q(y)||+1}{||Q(m)||}\right)\left(\frac{||Q(m)|| \ [\ ||Q(y)|| - 1]}{||Q(y)||+1}\right) = 1,\end{aligned}$$

and if $r \notin \overline{B}\left(0, \frac{||Q(y)||+1}{||Q(m)||}\right)$, then

$$||Q(y) - rQ(m)|| \geq \Big|\ |r|||Q(m) - ||Q(y)||\ \Big| >$$

$$\left(\frac{||Q(y)||+1}{||Q(m)||}\right)||Q(m)|| - ||Q(y)|| = 1.$$

Since $Q(m)$ was arbitrary, so $dist(Q(y), lin\{v\}) > 1$ for all $v \in V$. Because of the continuty of the natural map Q and since the vector x is an M-convex-cyclic vector for T, then there exists a convex polynomial $P(\alpha)$ such that

$$P(T)x - z \in Q^{-1}(v) \cap B(0, \varepsilon),$$

where $B(0,\varepsilon)$ is the open ball of X. In addition

$$dist\,(y, lin\{X_1, P(T)x\}) \geq dist\,(Q(y), lin\{Q(P(T)x)\}) > 1.$$

Finally, we can replace ε with $\frac{1}{n}$ for all $n \in \mathbb{N}$ and therefore find a desired sequence of convex polynomials and the proof is completed. □

Theorem 2.6. *Let $x \in X$ be a convex-cyclic vector for an operator $T \in L(X)$. then x is an M-convex-cyclic vector for a non-trivial closed subspace M of X.*

Proof. Consider closed ball $\overline{B}(0,1)$ of X and let $y \notin \overline{B}(0,1)$, then there exists a non-zero functional $\Lambda \in X^*$ with $||\Lambda|| = 1$, $\Lambda(y) = ||y||$. One can used the similar way in the proof of the Proposition 2.1 and construct one-dimensional subspace $M_1 \leq X$ such that $dist(y, M_1) > 1$. Since M_1

is one-dimensional, so consider A_1 as a countable dense subset of M_1. Break down the set of natural numbers $\mathbb{N}$ into disjoint union of infinite subsets N_k, $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $\{1, 2, \cdots, n\} \subset \bigcup_{k=1}^{n} N_k$ and let $A_1 = \{a_i\}_{i \in N_1}$. By the Proposition 2.1, for $k = 1$ we find a convex polynomial $P^1(\alpha)$ such that;

$$||P^1(T)x - a_1|| < \frac{1}{2} \qquad and \qquad dist\left(y, lin\{M_1, P^1(T)x\}\right) > 1.$$

Now set $M_2 = lin\{M_1, P^1(T)x\}$, $A_2 = \{a_i\}_{i \in N_2}$ is a countable dense subset of M_2. By similar way, for every $k \geq 2$, there exists a convex polynomial $P^k(\alpha)$ such that $||P^k(T)x - a_k|| < \dfrac{1}{2^k}$ and $dist\,(y, M_{k+1}) > 1$ where $M_{k+1} = lin\{P^k(T)x, M_k\}$ and $A_{k+1} = \{a_i\}_{i \in N_{k+1}}$ is a countable subset of M_{k+1}. Let $M = \overline{\bigcup_{k=1}^{\infty} M_k}$. Since $dist\,(y, M) \geq 1$, then M is a non-trivial closed subspace of X. Now let $z \in M$ be an arbitrary vector and $\varepsilon > 0$, then by density of the set $\bigcup_{i \geq 1} A_i$, for large enough k;

$$P^k(T)x \in B(z, \varepsilon),$$

therefore the vectore x is an M-convex-cyclic vector for the operator T and the proof is completed. $\square$.

3 Open Questions

The paper is ended with two questions on the subspace convex-cyclicity.

As it was pointed in the Introduction, every separable infinite-dimensional Banach space supports a hypercyclic operator. So every separable infinite-dimensional Banach space supports a subspace convex-cyclic operator. Now we want to delete the separability condition:

Question 1. Does every infinite-dimensional Banach space admit a subspace convex-cyclic operator which is not subspace hypercyclic?

Herrero in [8] established a spectral description of the closure of the set of hypercyclic operators on a Hilbert space. We ask a similar question for the subspace convex-cyclicity.

Question 2. Is there a spectral description of the closure of the set of subspace convex-cyclic operators on a Banach space?.

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